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Time-dependent solution of multidimensional Fokker–Planck equations in the weak noise limit

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Abstract. The time-dependent problem of multidimensional Fokker-Planck equations (FPE), satisfying potential conditions, is solved in the weak noise limit. For the relaxation from an intrinsically unstable state to the metastable state, we apply the theory of the Ω expansion of the Green function and reveal the possibility of dimensional reduction. From a metastable state to the stationary state, the time-dependent problem of the FPE is reduced to linear master equations. The first passage times are given explicitly.

1. Introduction

Dissipative systems under the influence of noise are often modelled by Fokker-Planck equations (FPE) [1-3]. In particular, the FPE is extensively used to describe systems far from equilibrium and has attracted much attention in recent years [2-6]. In most systems of practical interest, the noise represented by the diffusion term of the FPE can be considered as a small perturbation. Thus, the problem of how the systems by the FPE in the weak noise limit (or, say, the thermodynamic limit) turns out to be one of the most active fields of the last two decades.

In the study of the time-dependent problem of the FPE, the Ω expansion theory (ΩET) [5, 6] and the scaling theory (ST) [7, 8] are well known. The ΩET is successful in describing the evolution of the system from an extensive region to a stable state while failing in the region near an unstable point [5, 8, 9]. The ST is remarkable in characterising the evolution from a one-peak distribution to a two-peak one, starting from an unstable point while it has some trouble with the matching between various time regimes [7-9].

In [9-11] we suggested an approach of linearising the drift force in the initial time regime followed by the ΩET of the Green function to elucidate the whole process from an intrinsically unstable state to the metastable state (LΩEGF).

Nevertheless, as far as the time-dependent problem of the FPE is considered, we took into account, in all the references listed previously, only one-dimensional FPE. Apart from the simplest examples, e.g. the Ornstein-Uhlenbeck processes [12], the time-dependent problem of multidimensional FPE is poorly understood. Recently, several publications by Graham *et al* have considered the stationary solution of the multidimensional FPE without detailed balance in the weak noise limit [13-17]. The aim of the present paper is to extend the LΩEGF approach developed for the one-dimensional FPE to the multidimensional and multistable FPE.

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In this paper, we consider only the FPE with detailed balance, i.e. the FPE satisfying the potential conditions [18, 19]. Then the stationary solution of the FPE involved is assumed to be known explicitly. Throughout the presentation the weak noise limit is taken. In § 2 we apply the LΩEGF to the two-dimensional FPE and clarify the evolution from an intrinsically unstable state to the metastable state. In § 3, the first passage time and the process from a metastable state to the final stationary state is elucidated. The last section extends the theory to general q-dimensional FPE.

2. From an unstable state to the metastable state

2.1. Model

We first study the two-dimensional FPE in detail. Assume that the system can be modelled by the FPE

$$\partial p(x, y, t)/\partial t = -(\partial/\partial x)[c_1(x, y)p(x, y, t)] - (\partial/\partial y)[c_2(x, y)p(x, y, t)] + (\varepsilon/2)(\partial^2/\partial x^2 + \partial^2/\partial y^2)p(x, y, t) \quad (2.1)$$

where all the parameters, as well as the variables, are assumed to be dimensionless after a suitable rescaling. Moreover, we have

$$\varepsilon \ll 1.$$

For the sake of simplicity and clarity, the diffusion matrix is set to unity. Since the FPE satisfies the potential conditions, the planar vector should be constrained by

$$\partial c_1/\partial y = \partial c_2/\partial x. \quad (2.2)$$

Thus, the stationary solution of (2.1) can be worked out as

$$p(x, y) = N \exp[-u(x, y)/\varepsilon] \quad (2.3)$$

with N being a normalisation constant and $u(x, y)$ being given by

$$\partial u/\partial x = -2c_1 \qquad \partial u/\partial y = -2c_2. \quad (2.4)$$

It is well known that $u(x, y)$ decreases monotonically:

$$\mathrm{d}u(x, y)/\mathrm{d}t < 0 \quad (2.5)$$

as it is evolved by the corresponding deterministic equations:

$$\dot{x} = c_1(x, y) \qquad \dot{y} = c_2(x, y). \quad (2.6)$$

Hence, $u(x, y)$, the so-called potential, takes extreme values at all the singular points

$$c_1(x_i, y_i) = c_2(x_i, y_i) = 0 \qquad i = 1, 2, \ldots, n.$$

In particular, it takes minimal values at the attracting points of (2.6) and maximal values at the repellors. Moreover, the potential has a local saddle about the saddles of (2.6). In this section we study the initial-value problem of a system, starting from a delta function about the origin, an unstable point of (2.6),

$$p(x, y, 0) = \delta(x - a\sqrt{\varepsilon})\delta(y - b\sqrt{\varepsilon}) \quad (2.7)$$

with a and b being finite.

2.2. The LΩEGF

In the initial time regime, a major portion of probability centres on the vicinity of the origin. Thus, the drift force of (2.1) can be linearised. Moreover, the two directions of the eigenvectors of the linear part of the vector field can be set, after a suitable transformation, along the x and y axes, respectively. Finally, the FPE with linearised drift is

$$\partial p(x, y, t)/\partial t = -(\partial/\partial x)[\lambda_1 x p(x, y, t)] - (\partial/\partial y)[\lambda_2 y p(x, y, t)] + (\varepsilon/2)(\partial^2/\partial x^2 + \partial^2/\partial y^2)p(x, y, t) \qquad \lambda_1, \lambda_2 > 0. \tag{2.8}$$

The solution of the Ornstein–Uhlenbeck process can be directly written down as

$$p(x, y, t) = (\lambda_1\lambda_2/\{\pi^2\varepsilon^2[1-\exp(2\lambda_1 t)][1-\exp(2\lambda_2 t)]\}^{1/2} \times \exp\{\lambda_1[x - a\sqrt{\varepsilon}\exp(\lambda_1 t)]^2/\varepsilon[1-\exp(2\lambda_1 t)] + \lambda_2[y - b\sqrt{\varepsilon}\exp(\lambda_2 t)]^2/\varepsilon[1-\exp(2\lambda_2 t)]\}. \tag{2.9}$$

In the initial time regime

$$\exp(2\lambda_1 t), \exp(2\lambda_2 t) \ll 1/\varepsilon \tag{2.10}$$

the solution (2.9) is a good approximation of the actual evolution of (2.1). In the case of

$$\exp(2\lambda_1 t), \exp(2\lambda_2 t) \gg 1 \tag{2.11}$$

most of the probability flows out of the unstable region

$$x^2 + y^2 = O(\varepsilon)$$

and then the ΩET of the Green function is desirable [9, 11]. Since $\varepsilon \ll 1$, we may readily find a suitable time t_s satisfying

$$1/\varepsilon \gg \exp(2\lambda_1 t_s), \exp(2\lambda_2 t_s) \gg 1 \tag{2.12}$$

when the linearisation of the drift, as well as the ΩET of the Green function, hold simultaneously. Provided the probability distribution at time t_s is a delta function

$$p(x, y, t_s) = \delta(x - x_s)\delta(y - y_s) \tag{2.13a}$$

$$|x_s|, |y_s| \gg \varepsilon. \tag{2.13b}$$

(The condition (2.13*b*) guarantees that the distribution is far away from the unstable point and the ΩET works.) The solution given by the ΩET is

$$p(x, y, t) = \exp\{-[T - T(t)]\sigma^{-1}[T' - T(t)']/2\varepsilon\}[2\pi\varepsilon|\det(\sigma)|^{1/2}]^{-1} \tag{2.14}$$

where T and $T(t)$ are two-dimensional vectors:

$$T = (x, y) \qquad T(t) = (x(t), y(t)) \tag{2.15}$$

and T, $T(t)$ are their transpositions, respectively. σ is a matrix:

$$\sigma = \begin{pmatrix} \sigma_{xx}(t) & \sigma_{xy}(t) \\ \sigma_{xy}(t) & \sigma_{yy}(t) \end{pmatrix} \tag{2.16}$$

of which the elements can be solved from

$$\begin{aligned} \dot\sigma_{xx} &= (2\partial c_1/\partial x)\sigma_{xx} + 1 \\ \dot\sigma_{xy} &= (\partial c_1/\partial y)\sigma_{yy} + (\partial c_2/\partial x)\sigma_{xx} + (\partial c_1/\partial x + \partial c_2/\partial y)\sigma_{xy} \\ \dot\sigma_{yy} &= (2\partial c_2/\partial y)\sigma_{yy} + 1 \end{aligned} \tag{2.17}$$

with

$$x = x(t) \qquad y = y(t)$$

and $x(t), y(t)$ being given by

$$\begin{aligned} \dot{x}(t) &= c_1[x(t), y(t)] \\ \dot{y}(t) &= c_2[x(t), y(t)]. \end{aligned} \tag{2.18}$$

The initial conditions for (2.18) and (2.17) are

$$\begin{aligned} &x(t_s) = x_s \qquad y(t_s) = y_s \\ &\sigma_{xx}(t_s) = \sigma_{yy}(t_s) = \sigma_{xy}(t_s) = 0. \end{aligned} \tag{2.19}$$

Taking (2.9) as the initial distribution of (2.14) at t_s, we have $p(x, y, t) = (\lambda_1\lambda_2/\{4\pi^4\varepsilon^4[1-\exp(2\lambda_1 t_s)][1-\exp(2\lambda_2 t_s)][|\det(\sigma)|]\})^{1/2}$

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\mathrm{d}x_s\,\mathrm{d}y_s \exp\{\lambda_1[x_s - a\sqrt{\varepsilon}\exp(\lambda_1 t_s)]^2/\varepsilon(1-\exp(2\lambda_1 t_s) \\ &+\lambda_2[y_s - b\sqrt{\varepsilon}\exp(\lambda_2 t_s)]^2/\varepsilon(1-\exp(2\lambda_2 t_s) \\ &-[T-T(t)]\sigma^{-1}[T-T(t)]/2\varepsilon\}. \end{aligned} \tag{2.20}$$

According to the previous discussion, the actual solution of (2.1) can be represented by (2.9) as $t < t_s$ and (2.20) as $t > t_s$. It is interesting to point out that the integral (2.20) enjoys a remarkable property. It is right not only in the time regime $t > t_s$ but also in the initial time regime $t < t_s$. In [11], in the one-dimensional case, we have shown, by careful verification, that the integral (2.20) leads to (2.9) as $t < t_s$. In the two-dimensional case, the calculation procedure and the conclusion are exactly the same. In fact, the distribution given by (2.20) even turns back into the delta function (2.7) as $t - t_s \to -t_s$.

Thus, equation (2.20) provides the evolution of (2.1) from an intrinsically unstable state to the metastable state in the weak noise limit. The problem of solving two-dimensional FPE is reduced to the much simpler problem of solving the corresponding ordinary differential equations with the same dimension.

2.3. Reduction of dimension

Around the unstable state, at the origin, there might be several potential wells. According to (2.20), different potential wells may eventually gain different amounts of probability. A problem of practical importance is how much probability each potential well eventually obtains. It seems that there is no direct answer other than the detailed calculation of (2.20), which is still not easy, being completed. Fortunately, equation (2.20) may enjoy a further substantial simplification in the limit $\varepsilon \ll 1$.

Let us suppose $\lambda_1 > \lambda_2$. Hence, the probability diffusion along the x axis must be faster than along the y axis. As t increases, the ratio of the diffusion distance in the x direction to that in the y direction becomes larger. In the case of

$$1/\varepsilon \gg \exp(2\lambda_1 t) \gg \exp(2\lambda_2 t), 1 \tag{2.21}$$

the evolution is still in the initial time regime while the probability distribution turns out to be a very long thin strip along the x axis. In contrast, the extension of the distribution along the y axis is rather narrow, compared with the x axis, and can be neglected. Thus, the distribution, to a good approximation, can be regarded as one-dimensionally distributed. Actually, the distribution will follow the most unstable manifold. Moreover, out of the linear regime, the ΩET plays its role and then the probability peak will follow the deterministic path. This path must be the most unstable manifold since the preparation of the distribution in the linear regime is on that manifold.

Now the answer to the previous question is clear. So long as the distribution starts from an intrinsically unstable state (i.e. a and b in (2.7) are finite or zero), only the potential wells (at most, two) directed by the most unstable manifold, that is tangent to the x axis at the origin, may eventually gain finite amounts of probability. The probabilities obtained by all other potential wells are negligibly small. The final probability distribution is almost irrelevant to the initial coordinate b. However, the parameter a is important for the metastable state. As $t \to \infty$, the amount of probability

$$p^+ = \int_{-a}^{\infty} \mathrm{d}v \exp(-v^2/2)/\sqrt{2\pi} \tag{2.22a}$$

is contained by the potential well directed by the most unstable manifold of the positive x axis, and the remaining amount

$$p^- = \int_{-\infty}^{-a} \mathrm{d}v \exp(-v^2/2)/\sqrt{2\pi} \tag{2.22b}$$

is gained by the negative one.

Assuming that the most unstable manifold can be specified by

$$y = \phi(x) \tag{2.23}$$

we may further reduce (2.18) to a one-dimensional ordinary differential equation:

$$\dot{x}(t) = c_1\{(x(t), \phi[x(t)]\} \tag{2.24}$$

that can be explicitly solved, and then σ_{xx}, σ_{yy} follow (cf (2.17)). σ_{xy} may be provided in terms of $x(t)$, σ_{xx} and σ_{yy}. Finally, the integrand of (2.20) is given. The entire time-dependent problem from an unstable state to the metastable state is analytically settled. Usually, the problem of specifying the most unstable manifold is much easier than that of generally solving two-dimensional differential equations. Therefore, the reduction of dimension has a practical meaning. In particular, if the dimension q of the FPE is more than 2, the reduction of dimension from $q > 2$ to one may reduce greatly the difficulty of the calculations.

In some cases, the most unstable manifold can be readily presented, though an explicit solution of (2.18) is not available. Let us consider a two-box diffusion Schlog model. The drift may be specified as [20, 21]

$$\begin{aligned} c_1(x, y) &= rx - x^3 + d(y - x) \\ c_2(x, y) &= ry - y^3 + d(x - y) \end{aligned} \tag{2.55}$$

where the parameter d is introduced to represent the spatial diffusion. Here, we are not interested in the physical implication of the model, but focus our attention on solving the corresponding FPE. It is obvious that no explicit time-dependent solution of (2.18) can be found. The potential $u(x, y)$ is

$$u(x, y) = [rx^2 + ry^2 - x^4/2 - y^4/2 - d(x - y)^2]. \tag{2.26}$$

As $r > 3d$, the origin is an unstable point and there are four potential wells around the origin, which centre on the stable points

$$x_1 = y_1 = \sqrt{r} \tag{2.27a}$$

$$x_2 = y_2 = -\sqrt{r} \tag{2.27b}$$

$$x_3 = -y_3 = \sqrt{(r-2d)} \tag{2.27c}$$

$$x_4 = -y_4 = -\sqrt{(r-2d)} \tag{2.27d}$$

respectively. All these stable points can be connected with the origin by certain deterministic trajectories. About the origin, the linearisations of (2.18) are

$$\dot{w} = rw \qquad \dot{s} = (e-2d)s \tag{2.28}$$

with

$$s = \sqrt{2}(x-y)/2 \qquad w = \sqrt{2}(x+y)/2.$$

It is evident that the most unstable manifold is tangent to

$$s = x - y = 0 \tag{2.29}$$

at the origin. A remarkable property of (2.25) is that the curve (2.29) is invariant even in the non-linear region of (2.18). On the most unstable manifold (2.18) may be reduced to exactly

$$\dot{x}(t) = rx(t) - x(t) \qquad y(t) = x(t) \tag{2.30}$$

of which the solution is

$$\int_{x_s}^{x(t)} \mathrm{d}x/(rx - x^3) = t - t_s.$$

σ_{xx}, σ_{yy} and σ_{xy} can be written directly in terms of $x(t)$. Thus, the integrand of (2.20) is explicit.

The introduction of this fluctuation makes the FPE substantially different from the corresponding kinetic equations. Let us assume an experiment. Randomly release balls one by one in the $\sqrt{\varepsilon}$ vicinity of the origin. Each time the ball is evolved by the kinetic equations (2.6) (the vector field is assumed to be (2.25)) and may fall into one of four basins (cf (2.27)), according to the initial position. After many tests, one may certainly find that comparable amounts of balls have been received by all the four wells. In contrast, doing the same thing while replacing the balls and the kinetic equations by various delta functions and the FPE, respectively, one may find almost nothing in the wells $x = -y = \pm\sqrt{(r-2d)}$. A major portion of probability is gained by the wells $x = y = \pm\sqrt{r}$. In fact, the final probability distribution is

$$p(s, w, \infty) = N \exp(-s^2/2\varepsilon\beta_2)\{\exp[-(w-\sqrt{r})^2/2\varepsilon\beta_1] + \exp[-(w+\sqrt{r})^2/2\varepsilon\beta_1]\} \tag{2.31}$$

with

$$\beta_1 = \sigma_{ww} = \sqrt{2}r/2(2r-d)^2 \qquad \beta_2 = \sigma_{ss} = (r-d)\beta_1/r$$

and

$$N = 1/4\pi\varepsilon(\beta_1\beta_2)^{1/2}.$$

The total probability is eventually equally divided by the two basins $x = y = \pm r$.

If the initial delta function is prepared as

$$a=0 \qquad |b| \neq 0$$

according to the kinetic equations, the state finally realised must be the stable state directed by the invariant manifold tangent to the y axis. However, according to the FPE, only the stable states directed by the manifold tangent to the x axis can be realised as $\lambda_1 > \lambda_2 > 0$. This highly unexpected behaviour is caused by the fluctuation.

Now an essential point should be mentioned. In (2.20), there are two unstable modes about the unstable point (the origin), the most unstable mode plays a key role. It wins the competition in the vicinity of the unstable point and then completely governs the system in the subsequent evolution. It leads to the reduction of the dimension involved. This intuitive picture reminds us the slaving principle in synergetics theory [2, 3]. However, there are two substantial differences between these two cases. First, the slaving principle emphasises that the slow unstable modes control fast stable modes. In our case all modes are unstable. Our conclusion is that the most unstable mode controls other unstable modes (as well as stable modes) and determines the destiny of the FPE system. Second, so far as the adiabatic elimination of the fast modes is performed, one usually treats the evolution slightly over a bifurcation threshold, $0 < \lambda_i \ll 1$, where λ_i are the eigenvalues of the unstable modes. In the present treatment, we are able to deal with the situations far beyond the threshold.

2.4. *Several scaling relations*

In § 2.3, we predicted that, starting from an intrinsically unstable state, a major portion of probability must flow asymptotically into the wells directed by the most unstable manifold of the unstable point. How can one alter this destination and distribute comparable probabilities into other wells?

Consider a probability peak initially located at the origin. Following (2.20), the probability distribution can develop to a distance $\sqrt{\varepsilon}\exp(\lambda_1 t)$ away along the x axis from the origin. As

$$\exp(\lambda_1\tau) = O(1/\sqrt{\varepsilon}) \qquad \text{or} \qquad \tau = -\ln \varepsilon/2\lambda_1 \tag{2.32}$$

the diffusion distance along the x axis reaches a macroscopic quantity. In order that the diffusion along the y axis reaches a macroscopic quantity as well as τ (so that the probability may be comparably distributed in the various wells), we require

$$\exp(\lambda_2\tau) = O(1/\sqrt{\varepsilon}) \qquad \text{or} \qquad \exp[(\lambda_1-\lambda_2)\tau] = O(1). \tag{2.33}$$

Thus, in the case of

$$\lambda_1 - \lambda_2 \gg -2\lambda_1/\ln \varepsilon \tag{2.34}$$

the initial value problem of the FPE may enjoy the reduction of dimension from two to one. In the opposite case

$$\lambda_1 - \lambda_2 \ll -2\lambda_1/\ln \varepsilon \tag{2.35}$$

the difference $\lambda_1 - \lambda_2$ plays no role and the problem may be treated as if $\lambda_1 = \lambda_2$. In the intermediate situation

$$\lambda_1 - \lambda_2 = O(-2\lambda_1/\ln \varepsilon) \tag{2.36}$$

all the wells connected with the origin by certain invariant manifolds may gain comparable probabilities and meanwhile the difference $\lambda_1 - \lambda_2$ does influence the distribution of probability in various potential wells. In all the last two cases, the reduction of dimension fails.

Besides (2.36), there is one more scaling relation. Provided the initial probability peak is such that $a = 0, b \neq 0$, at time $\tau = -(1/2\lambda_1) \ln \varepsilon$ the probability distribution flows up to a macroscopic distance along the x axis. In order that finite quantities of probability are eventually obtained by the wells directed by the less unstable manifold, it is required that

$$b\sqrt{\varepsilon}\exp(\lambda_2\tau) = O(\exp(\lambda_1\tau))$$

leading to

$$b = O(\varepsilon^{-(\lambda_1-\lambda_2)/2\lambda_1})$$

or

$$y_0 = b\sqrt{\varepsilon} = O(\varepsilon^{\lambda_2/2\lambda_1}). \tag{2.37}$$

Thus, we have

$$1 \gg y_0 \gg \sqrt{\varepsilon}$$

in the case that

$$\lambda_1 > \lambda_2 > 0 \qquad \lambda_1 - \lambda_2 = O(1) \qquad \lambda_2 = O(1)$$

are satisfied. Under the condition

$$b \ll \varepsilon^{-(\lambda_1-\lambda_2)/2\lambda_1} \tag{2.38}$$

the reduction of dimension is desirable and the most unstable manifold is prevalent; under the opposite condition

$$b \gg \varepsilon^{-(\lambda_1-\lambda_2)/2\lambda_1} \tag{2.39}$$

the fluctuation plays no role and the system can be regarded as completely evolved by the deterministic equations. In the intermediate situation (2.37), one can make neither the reduction of dimension nor the neglect of the fluctuation, and an essential two-dimensional fluctuated system should be taken into account. The scaling relations (2.34) and (2.38) provide the conditions under which the reduction of dimension can be enjoyed.

3. From a metastable state to the stationary state

Equation (2.20) represents the evolution of the probability distribution, governed by the FPE, from $t = 0$ to $t \to \infty$. However, the state asymptotically realised is merely a metastable state. In the present section we will study the process from a metastable state to the final stationary state, i.e. study the probability transitions between various potential wells.

In order to calculate the transition rates between different deterministic stable states, let us keep in mind that the time needed for the evolution from an unstable state to the metastable state (order of $-\ln \varepsilon$) is negligibly short, in comparison with the time for the metastable state (order of $\exp(1/\varepsilon)$). It leads to an assumption: in a metastable state, the density distributions in each potential well are proportional to the stationary probability distribution. Thus, in the entire time regime of the metastable state, the stationary distribution is kept unchanged locally, though the probability transitions from one well to another do proceed all the time so long as the probability balance between various basins are not established. Moreover, probability may be transported from one basin to another only by crossing the potential barrier between them. Our second assumption is that, in the weak noise limit, the probability transition between any two wells is mainly due to the saddle, connected with both wells, for which the potential is minimal along the entire barrier. The reason for the second assumption is transparent. Provided $u(D)$ is the potential of the given saddle while $u(R)$ is that of any point on the barrier, the conditions $u(D) < u(R)$ and $\varepsilon \ll 1$ lead to

$$P_R = N \exp[-u(R)/\varepsilon] \ll P_D = N \exp[-u(D)/\varepsilon].$$

Therefore, in comparison with the probability transition through the vicinity of the saddle, those through other parts of the barrier are negligibly small.

3.1. The first passage time

Assume again that the origin is at the saddle point, for which the potential is minimal along the barrier between two wells A (in region $x > 0$) and B (in region $x < 0$). Without losing generality we further assume that the unstable manifold of the saddle is along the x axis and the stable one is along the y axis. The initial distribution is such that B is filled by an amount of probability $p(t)$ while A is empty. Let us study the probability transition flow from B to A.

About the saddle, the FPE (2.1) can be reduced to

$$\begin{aligned}\partial p(x, y, t)/\partial t = &-(\partial/\partial x)[\eta_1 x p(x, y, t)] + (\partial/\partial y)[\eta_2 y p(x, y, t)] \\ &+ (\varepsilon/2)(\partial^2/\partial x^2 + \partial^2/\partial y^2) p(x, y, t)\end{aligned} \tag{3.1}$$

with

$$\eta_1, \eta_2 > 0$$

and the initial probability distribution takes the form

$$p(x, y, t_0) = \begin{cases} N(t_0) \exp[(\eta_1 x^2 - \eta_2 y^2)/\varepsilon] & x < 0 \\ 0 & x > 0 \end{cases} \tag{3.2}$$

where the constant $N(t_0)$ is related to $p_B(t_0)$. Equation (3.1) can be solved as

$$\begin{aligned} p(x, y, t) = &(\eta_1 \eta_2)^{1/2} (\pi\varepsilon\{\exp[2\eta_1(t - t_0)] - 1\}\{1 - \exp[-2\eta_2(t - t_0)]\})^{1/2} \\ &\exp[\eta_1\{x - x_0 \exp[\eta_1(t - t_0)]\}^2/(\varepsilon\{1 - \exp[2\eta_1(t - t_0)]\})] \\ &\exp \times [\eta_2\{y - y_0 \exp[-\eta_2(t - t_0)]\}^2/(\varepsilon\{\exp[-2\eta_2(t - t_0)] - 1\})] \end{aligned} \tag{3.3}$$

provided the initial distribution is

$$\bar{p}(x, y, t_0) = \delta(x - x_0)\delta(y - y_0). \tag{3.4}$$

Employing both (3.2) and (3.3), we have

$$p(x, y, t) = \int_{-\infty}^{0} \int_{-\infty}^{\infty} dx_0\, dy_0\, \bar{p}(x, y, t) p(x_0, y_0, t_0)$$

$$= N(t_0) \int_{-\infty}^{0} dx_0[\eta_1/(\pi\varepsilon\{\exp[2\eta_1(t-t_0)-1]\})]^{1/2} \exp(\eta_1 x_0^2/\varepsilon)$$

$$\times \exp[\eta_1\{x - x_0 \exp[\eta_1(t-t_0)]\}^2/(\varepsilon\{1-\exp[2\eta_1(t-t_0)]\})]$$

$$\times \int_{-\infty}^{\infty} dy_0[\eta_2/(\pi\varepsilon\{1-\exp[-2\eta_2(t-t_0)]\})]^{1/2} \exp(-\eta_2 y^2/\varepsilon)$$

$$\times \exp[\eta_2\{y - y_0 \exp[-\eta_2(t-t_0)]\}^2/(\varepsilon\{\exp[-2\eta_2(t-t_0)]-1\})]. \quad (3.5)$$

In the time interval

$$\exp(-1/\varepsilon) \gg t - t_0 \gg 1 \quad (3.6)$$

equation (3.5) gives rise to

$$p(x, y, t) = N(t_0)(\eta_1/\pi\varepsilon)^{1/2} \exp(-\eta_2 y^2/\varepsilon) \int_{-\infty}^{0} (dx_0/\sqrt{\pi}) \exp(-x_0^2 + 2\sqrt{\eta_1} x x_0/\sqrt{\varepsilon}). \quad (3.7)$$

In the exponent of the integrand of (3.7) we neglect the term of x^2 since only the behaviour close to the origin is relevant to the probability transition. A remarkable feature of (3.7) is that after a long enough time ($t - t_0 \gg 1$) the discontinuity of the distribution at $t = t_0$ on $x = 0$ is ruled out. However, the time is still short enough ($t - t_0 \ll \exp(1/\varepsilon)$) to keep the amount of probability in B unchanged ($N(t_0)$ can be replaced by $N(t)$). One more feature of (3.7) is that the distribution is independent of t. The result is reasonable as well as instructive. After a short transient process, the density distribution as well as the flow about the saddle turns out to be stable. It is just what happens for a liquid flow from a large vessel though a small hole. Indeed this is a good picture for the FPE in the weak noise limit. Hence, in the following we shall use t instead of t_0.

Inserting (3.7) into (3.1) and integrating both sides of (3.1) over $-\infty < x < 0$, and $-\infty < y < \infty$, we have

$$dp_B(t)/dt = -(\varepsilon/2)\, d\left(\int_{-\infty}^{\infty} dy\, p(x, y, t)\right)(dx_{|x=0})^{-1} \quad (3.8)$$

which yields

$$dp_B(t)/dt = -N(t)(\eta_1\varepsilon/\pi)^{1/2}/2. \quad (3.9)$$

In the weak noise limit, the probability distribution in basin B can be well approximated by a Gaussian distribution

$$p_B(t) = M(t) \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dw \exp[-u(B)/\varepsilon - \beta_1(B)s^2/\varepsilon - \beta_2(B)w^2/\varepsilon]$$

$$= \pi M(t)\varepsilon \exp[-u(B)/\varepsilon]/(\beta_1(B)\beta_2(B))^{-1/2} \quad (3.10)$$

where $\beta_1(B)$ and $\beta_2(B)$ are the two eigenvalues of the linear part of the drift on B. Comparing (3.2) and (3.10) we have

$$N(t) = M(t) \exp[-u(0,0)/\varepsilon] = (\beta_1(B)\beta_2(B))^{1/2} \exp[(u(B) - u(0,0))/\varepsilon] p_B(t)/\pi\varepsilon. \quad (3.11)$$

Finally, equation (3.9) is specified as

$$\begin{aligned} &\mathrm{d}p_{\mathrm{B}}(t)/\mathrm{d}t = -R_{\mathrm{BA}}p_{\mathrm{B}}(t) \\ &R_{\mathrm{BA}} = [\beta_1(\mathrm{B})\beta_2(\mathrm{B})\eta_1/4\pi^3\varepsilon]\exp[(u(B)-u(0,0)]/\varepsilon]. \end{aligned} \tag{3.12}$$

In general, both potential wells A and B may contain probability simultaneously. Therefore, equation (3.12) should be generalised to

$$\begin{aligned} &\mathrm{d}p_{\mathrm{B}}(t)/\mathrm{d}t = -R_{\mathrm{BA}}p_{\mathrm{B}}(t) + R_{\mathrm{AB}}p_{\mathrm{A}}(t) \\ &dp_{\mathrm{A}}(t)/\mathrm{d}t = -R_{\mathrm{AB}}p_{\mathrm{A}}(t) + R_{\mathrm{BA}}p_{\mathrm{B}}(t) \end{aligned} \tag{3.13}$$

where R_{AB} is exactly the same as R_{BA} with $\beta_1(\mathrm{B}), \beta_2(\mathrm{B})$ and $u(\mathrm{B})$ replaced by $\beta_1(\mathrm{A}), \beta_2(\mathrm{A})$ and $u(\mathrm{A})$, respectively. The mean first passage time is given by the Kramers escape rates as

$$T_{\mathrm{AB}} = 1/(\mathrm{R}_{\mathrm{AB}} + R_{\mathrm{BA}}). \tag{3.14}$$

It is, indeed, of order $\exp(1/\varepsilon)$. It is worth remarking that T_{AB} is irrelevant to $-\eta_2$, the negative eigenvalue of the linear part of the vector field on the saddle.

The problem of the first passage time has been discussed extensively (cf [1] and references therein). Here, we extend the study to the two-dimensional problem based on the two intuitive assumptions. In case of the one-dimensional FPE our result (3.12) and (3.14) recovers the well known form of the first passage time.

3.2. Probability balance equations

Suppose kinetic equations (2.6) have n attractors, $A, \ldots, A$. The potential must have n basins, each centred on an attractor. Then the general probability balance equations are

$$\mathrm{d}p(A_i, t)/\mathrm{d}t = -\sum_{j=1}^{n} R_{ij}p(A_i, t) + \sum_{j=1}^{n} R_{ji}p(A_j, t). \tag{3.15}$$

R_{ij} is given by (3.12) with $u(0,0)$ replaced by $u(D_{ij})$. D_{ij} is the saddle connecting both A_i and A_j, and has minimal potential among all the saddles of such kind. R_{ij} is set to be zero if no saddle connects A_i and A_j.

Equations (3.15) are linear master equations which can be analytically solved or discussed. Based on (2.20) and (3.15), the evolution of the system, governed by the FPE, from an intrinsically unstable state up to the final stationary state can be entirely clarified.

4. The extension to q-dimensional FPE

An extension of all the previous procedures to the q-dimensional problem is direct.

Suppose the FPE

$$\begin{aligned} &\partial p(\boldsymbol{x}, t)/\partial t = -\sum_{\lambda=1}^{q} (\partial/\partial x_i)[c_i(\boldsymbol{x})p(\boldsymbol{x}, t)] + (\varepsilon/2)\sum_{i=1}^{j} (\partial^2/\partial x_i^2)p(\boldsymbol{x}, t) \\ &\boldsymbol{x} = (x_1, x_2, \ldots, x_q) \end{aligned} \tag{4.1}$$

has a stationary probability distribution

$$\begin{aligned} &p(\boldsymbol{x}) = N\exp[-u(\boldsymbol{x})/\varepsilon] \\ &\partial u(\boldsymbol{x})/\partial x_i = -2c_i(\boldsymbol{x}). \end{aligned} \tag{4.2}$$

the origin is assumed to be an extremum of the potential. Moreover, the linearisations of the kinetic equations about the origin have the forms of

$$\dot{x}_i = \lambda_i x_i \tag{4.3}$$

with

$$\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_q \qquad \lambda_1 > 0.$$

The initial probability distribution is

$$p(\boldsymbol{x}, 0) = \prod_{i=1}^{q} \delta(x_i - a_i\sqrt{\varepsilon}).$$

Now equation (2.20) can be generalised to

$$p(\boldsymbol{x}, t) = \left(\prod_{i=1}^{q} \lambda_i/\{\pi\varepsilon[\exp(2\lambda_i t) - 1]\}\right)^{1/2} (1/\{2\varepsilon|\det[\sigma(t)]|\})^{1/2}$$
$$\times \exp\left[\left(\sum_{i=1}^{q} (-\lambda_i[x_{is} - a_i\sqrt{\varepsilon}\exp(\lambda_i t_s)]^2/\{\varepsilon[\exp(2\lambda_i t_s) - 1]\})\right)\right.$$
$$\left. - [\boldsymbol{x} - \boldsymbol{x}(t)]\sigma^{-1}(t)[\boldsymbol{x}' - \boldsymbol{x}(t)']/2\varepsilon\right] \prod_{i=1}^{q} \mathrm{d}x_i \tag{4.4}$$

where $\boldsymbol{x}'$ and $\boldsymbol{x}(t)'$ are the transpositions of $\boldsymbol{x}$ and $\boldsymbol{x}(t)$, respectively. $\boldsymbol{x}(t)$ can be solved from the deterministic equations

$$\dot{x}_i(t) = c_i[\boldsymbol{x}(t)] \tag{4.5}$$

with the initial conditions

$$x_i(t_s) = x_{is} \qquad i = 1, 2, \ldots, q.$$

The elements of the matrix $\sigma(t)$ are given by

$$\dot{\sigma}_{ii} = 2\partial c_i/\partial x_i \sigma_{ii} + 1$$
$$\dot{\sigma}_{ij} = \partial c_i/\partial x_j \sigma_{jj} + \partial c_j/\partial x_i \sigma_{ii} + (\partial c_i/\partial x_i + \partial c_j/\partial x_j)\sigma_{ij} \qquad i > j$$

whence the deterministic equations (4.5) are solved and the matrix follows. Consequently, the integrand of (4.4) is explicit. Therefore the problem of solving a q-dimensional FPE is reduced to that of solving q-dimensional ordinary differential equations. The latter is, of course, much easier than the former.

Starting from an intrinsically unstable state (i.e. $a_i = \mathrm{O}(1)$, $i = q, 2, \ldots, q$), the problem may be further simplified, as stated in § 2.3. Up to the metastable state, the system is controlled by the most unstable manifold and the dimension of the kinetic equations involved can be reduced from q to one! On the most unstable manifold, we have

$$x_i(t) = f_i[x_1(t)] \qquad i = 2, \ldots, q.$$

The trajectory is produced by

$$\mathrm{d}x_1(t)/\mathrm{d}t = c_i[x_1(t), f_2(x_1), f_3(x_1), \ldots, f_q(x_1)] \tag{4.6}$$

which yields the integrand of (4.4) analytically.

For q-dimensional FPE, we can also obtain two kinds of scaling relations analogous to (2.34) and (2.38) under which the reduction of dimension from q to one is available. Here we do not repeat the similar description.

For the evolution from a metastable state to the stationary distribution, all equations (3.9)-(3.15) can be used except that R_{ij} is replaced by

$$R_{ij} = \left(\prod_{\nu=1}^{q} [(\beta_\nu(A_i)/(2\pi\varepsilon)] \right)^{1/2} (\eta_1 \varepsilon/\pi)^{1/2} \exp\{[u(A_i) - u(D_{ij})]/\varepsilon\} \tag{4.7}$$

with η_1 being the positive eigenvalue of the linear part of the vector field on the saddle D_{ij} which connects the two stable points A_i and A_j and has the minimal potential along the barrier between the two wells.

In this paper, only the FPE with detailed balance is considered. The extension of the present approach to the FPE with non-zero circulation will introduce some essentially new points. We will study these matters in future papers.

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